Instructional qualities of a successful mathematical problem-solving class*

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Problem solving has been an important issue in mathematics education during the last 25 years. Research studies in this area have documented extensively what students show while working with mathematical problems (resources, beliefs, and cognitive and metacognitive strategies). However, there is little information about how mathematical problem-solving activities should be implemented in regular classrooms. This paper shows what activities have been successful implemented by an expert during a mathematical problem-solving course. It focuses on the identification of qualities of the problems used to promote the development of students strategies and values that reflect mathematical practice in the classroom. Special attention is paid to the type of question that help students evaluate their own problem-solving processes and the importance of allowing the students to work in small group activities. Although the course itself does not address a particular sequence of content (algebra, geometry or calculus), there is indication that it could be used as a model to be adjusted to a regular mathematical course.

1. Introduction

Recent proposals in mathematical problem-solving instruction have suggested that students in their learning experiences should be engaged in activities that are related to the practice of doing mathematics. As Schoenfeld pointed out '... doing mathematics is fundamentally an act of sense-making, an act of taking things apart (mathematically) and seeing what makes them tick' [1, p. 87]. Thus, learning mathematics goes beyond studying rules, procedures or algorithms, it involves the use of both heuristics and metacognitive strategies to solve problems, the use of various representations to make sense of information, and the search for mathematical connections or applications in different contexts. Here, it becomes important that students develop a mathematical disposition, and a set of beliefs and attitudes consistent with the practice of doing mathematics. What type of learning activities tend to promote mathematical values during the implementation of problem-solving instruction? What kind of tasks help students to engage in mathematical discussion in the classroom? What type of evaluation could be used to assess students’ progress in mathematical problem-solving? These are issues

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that need to be addressed in order to evaluate the potential use of mathematical problem-solving instruction.

An important stage in the research on problem-solving has been to document what students do while working on mathematical tasks. For example, Schoenfeld [2] found that the process of doing mathematics includes the use of resources or basic mathematical knowledge (facts, procedures, algorithms), the use of heuristic strategies, the presence of metacognitive activities (monitoring and control), and an understanding of the nature of the mathematical practice (conception of the discipline). As a consequence, it is necessary to investigate to what extent the students' problem-solving behaviours could be improved when the instruction they receive takes into account learning activities related to those dimensions.

The purpose of this paper is to discuss aspects related to the implementation of problem-solving activities in the classroom. The discussion will be based on the analysis of segments from a mathematical problem-solving course taught at university level. This course has been part of a research programme in problem-solving for about 20 years. The results have shown that students who take the course make significant progress in the development of their problem solving abilities [2–4]. Thus, in this paper specific components of the course are discussed that illustrate what and how some instructional activities are implemented. There will be special attention to the identification of mathematical messages or morals associated with the problems used during the course.

2. Background to mathematical problem-solving instruction

Mathematical problem solving has been part of the mathematics education arena for several years. Polya [5] provides a general framework that has been used to propose learning activities for the classroom. Stanic and Kilpatrick [6, p. 1] stated that 'the term problem solving has become a slogan encompassing different views of what education is, of what schooling is, of what mathematics is and of why we should teach mathematics in general and problem-solving in particular'. Indeed, mathematics curriculum, instructional methods, and research programmes in problem solving could differ in terms of principles, general goals or instructional focus. Although there might be different directions in what aspects of problem solving to investigate, there seems to be an agreement about the importance of engaging students in problem-solving experiences. The National Council of Teachers of Mathematics [7] has pointed out that mathematical problem solving should be an important activity in the learning of mathematics; however, the need to investigate the effects in the implementation of problem-solving activities continues to be a priority in the research agenda [8]. In addition, although there are general principles associated with mathematical problem-solving instruction such as the importance of relating the practice of doing mathematics to the process of learning it and the active participation of the students during the class activities, there is no unique way to implement learning activities consistent with this approach. As a consequence, it is important to document what type of implementation produces what type of results in the students’ learning of mathematics.
3. Themes and approaches to mathematical problem-solving instruction

The idea that a problem-solving course may focus on various aspects of mathematical practice (heuristics, resources, metacognition, etc.) during the implementation period, makes it important to identify some features related to several approaches. For example, in the early 1980s several of the problem-solving approaches relied on Pólya's four-phase model as the main structure of the class. Later, there was more attention to the role of non-routine problems and the presence of metacognitive strategies as an important component of instruction. Here, an important line of research was to study fundamental differences between mathematicians and students while working on mathematical problems (expert–novice studies). Recently there has been interest in considering the role of social factors during the learning of mathematics. 'Learning occurs as people engage in activities and the meaning and significance of objects and information derive from their roles in the activities that people are engaged in' ([9, p. 100]. Indeed, one important feature of Schoenfeld's problem-solving approach is that the students act as a mathematical community while working on mathematical tasks. However, few studies have analysed the effects of the implementation of these types of instruction [8].

Although it may be possible to trace some of the general principles that distinguish an approach to learning mathematics based on problem solving, in general, there is little information about how these principles are implemented in the classroom and finally assimilated by the students. In this paper there is no intention to review differences between various mathematical problem-solving courses. Nevertheless, examples that illustrate the focus of some approaches might help us to recognize the difficulty in studying the implementation of problem solving in general terms. That is, it is necessary to discuss not only the general principles but also the specific problems and actions taken during the development of the course. For instance, some mathematics educators and practitioners who have been interested in problem solving have written about what ideas make this approach worthwhile considering during instruction. Three instructional approaches are listed below as a way to illustrate that a course in mathematical problem solving may include different learning activities during the implementation period. The first one focuses on the importance of studying a specific mathematical idea through the discussion of particular problems; these problems often are recognized as non-routine tasks. The second approach is an example in which the students directly reconstruct the main theorems associated with the course. The third approach focuses on the development of both cognitive and metacognitive students' strategies that could help students solve diverse types of problems. Some activities implemented during the development of the third approach (are the sources of data) will be analysed in this paper.

Schroeder and Lester [10] identify what they call teaching mathematics via problem solving as an approach to learning mathematics. The fundamental idea here is that a problem is used as a means to learn any mathematical content.

... problems are valued not only as a purpose for learning mathematics but also as a primary means of doing so. The teaching of a mathematical topic begins with a problem situation that embodies key aspects of the topic, and mathematical techniques are developed as reasonable responses to reasonable problems ([10] p. 33).
Schroeder and Lester [10] suggest that this approach could be used to study the formal content studied at school by selecting specific problems. During the discussion of such problems the students will have the opportunity to develop the mathematical content needed to solve the problems. In practice, given the rigidness and extension of the content to be covered in a regular course, some difficulties might arise during the selection of the type of problems that are used as a vehicle to introduce and study the content. In fact, a common variant that resembles features of this approach is that in which a problem is used as a motivation to introduce specific content to the class. Another interpretation of this approach might include that the content itself could be treated as a problematic situation to be discussed under the consideration of its representations, connections, and applications. Lester mentioned that teaching mathematics via problem solving has not been subjected to enough research scrutiny to make any claims about its potential [8, footnote, p. 666].

Halmos [11] presented another approach to problem solving to teach a linear algebra class. The material to be studied during the development of the course consisted of the main theorems that appear in a first linear algebra course. ‘The first day of class, I handed each student a set of 19 pages, ..., there were ... nothing but fifty theorems stated correctly but brutally, with no expository niceties’ [11, p. 853]. Thus, the theorems were the initial departure to discuss various strategies, to represent information and relationships, to look for means to prove them, and then to examine and extend possible connections. Regarding his role as an instructor during the class, Halmos stated:

I must not only be the moderator of what can easily turn into an unruly debate, but I must understand what is being presented, and when something fishy goes on I must interrupt with a firm but gentle ‘would you explain that please—I don’t understand.’ [11, p. 853]

It may be clear that Halmos’ students require a certain level of mathematical maturity to be engaged in the class discussion. Indeed, this course was offered to honours students who had shown some interest in pursuing mathematics in their careers. In relation to the success of the course, Halmos stated:

It worked. At the second meeting of class I said ‘O.K., Mr. Jones, let’s see you prove theorem 1,’ and I had to push and drag them along before they got off the ground. After a couple of weeks they were flying. They liked it, they learned from it, and they entered into the spirit of research—competition—..., glory, and all. [11, p. 853]

Schoenfeld, who has been teaching a mathematical problem-solving course for many years, does not address a content-oriented course directly; but throughout the development of the course, his students deal with several examples in which specific content is discussed. For example, the basic concepts of geometry (constructions), number theory, combinatorics, and calculus are topics that frequently appear as a context in the problem-solving class. It may be that the course helps students to reconceptualize their ideas about mathematics and deal with problems without focusing on specific content. However, that fact that basic mathematical ideas are addressed consistently during the course seems to suggest a new vision for the organization of mathematical curriculum. That is, rather than addressing a
specific sequence of content, it is important to deal with fundamental ideas of mathematics (including heuristics) that students could use to deal with problems from different areas and contexts. This idea is consistent with some curriculum proposals in which emphasis is given to the study of the essential or key mathematical concepts [12].

This paper focuses on documenting some events that appear consistently when using the third approach. Schoenfeld teaches the course at the university level. Perhaps, this course is among the few courses that have been attached to a research programme in mathematical problem solving. The fact that Schoenfeld himself teaches the course might offer some advantages during the implementation, however, the analysis and discussion of what happens during instruction could help other instructors to transfer or apply some of the activities related to the implementation of this approach. It is important to mention that the features or aspects of the problem-solving course addressed in this paper were identified by watching videotapes of the class development and observing the class directly.

4. Origin and basic principles associated with the course

To document some specific learning activities that consistently appear during the problem-solving course taught by Schoenfeld, it will be important to present some background associated with the course. Since the course has been implemented for about 20 years, there is special interest in tracing aspects that have been evolving throughout this period. For example, the detailed analysis by Schoenfeld of the students’ work during the course helped clarify the importance of meta-cognitive strategies [13]. Although Schoenfeld has written extensively about the changes in his course, it will be important to review what he shares with his students in his introduction to the course. This part is important to understand that a problem-solving course always offers room for changes and adjustments in accordance to what occurs during the class development.

Schoenfeld in his first day of classes outlines the main goals of the course and explains its key features. In his introduction to the course, he describes his own experience when he read Polya’s book *How to solve it*. Schoenfeld recognized that ideas presented by Polya are those that working mathematicians use in their work. Schoenfeld observed the potential of Polya’s ideas and in his introduction to the class, he says:

I finished an undergraduate career, I went through an entire career, as a graduate student, I’m a young professional, now for the first time I’m reading about these tricks of the trade. Why didn’t they tell me all that when I was a freshman and save me the trouble of discovering all of that for myself? Maybe it’s a version of the medieval trial by gauntlet, the only people who succeed without knowing the rules, I don’t know.

Schoenfeld mentions that his interest in Polya’s ideas resulted in a research programme in mathematical problem solving. One initial aim, in this programme was to explore how the work of Polya could help students in their learning of mathematics. In this period, the main focus of the course was to help students to use several heuristic strategies to solve mathematical problems. After several years of research and implementing the results in mathematical instruction, Schoenfeld is firm:
... one thing I can guarantee you is: It [Pólya’s work] does work. By the end of this course you will have an arsenal of problem-solving tools and techniques that will enable you to be much more successful, not only in solving problems that you’ve been shown how to solve, but also encountering new things and making sense of them which is something that your math courses don’t normally train you how to do.

As will be illustrated with various examples, a main emphasis of the course is the use of heuristic strategies to solve different types of problems. That is, there is explicit discussion among the students about what strategies to use at different stages of the solution processes.

Another important goal of the course is that students are engaged in the process of developing mathematics on their own. Schoenfeld believes that it is possible to create a class environment in which the students engage in activities that simulate what mathematicians do while working on mathematical ideas. ‘I work to make my problem-solving courses serve as microcosms of selected aspects of mathematical practice and culture—so that by participating in that culture, students may come to understand the mathematical enterprise in a particular way’ [4, p. 61]. In this context, the type of problems that students work on during the course involve various levels of difficulty. He tells his students:

... I’ll hand out problems, we’ll work on them in class, some of the problems can be solved fairly fast and some of them will merely serve as introduction to more conjectures and more problems. Other problems may be things that we visit for two or three days [classes] or even a week or two as we do something, find something interesting in it, but don’t make enough progress as a group, and we’ll say ‘fine, we’ll get back to it next time’ and we’ll keep working on the problem over a period of days. So what we do the vast majority of times in here is just do and talk mathematics.

An important activity of the course is that students have the opportunity to approach various problems and explore not only different solution methods but also pursue related conjectures and explicitly make connections to other areas or contexts. Thus an overarching goal in the course is that students learn to think mathematically. In general terms, the students are exposed to learning activities in which they are constantly asked to present their ideas, propose possible approaches, support and communicate their arguments, and evaluate and extend their solutions. So, it seems important to document how the students become engaged in these activities.

5. Features and qualities of the problems used during instruction

What makes the type of problems that are part of Schoenfeld’s problem-solving class interesting is not only the variety of mathematical ideas involved in the solution process, but also that the majority of the problems are accessible to the students. In general, the problems help students think of various questions that require mathematical resources and strategies in order to analyse them. In addition, the careful discussion of different methods of solution, connections to other situations, and extensions to more general cases are aspects in which students are exposed to new challenges. It seems that a basic property of the problems
chosen for the course is that they offer the opportunity for the students to become engaged in mathematical discussions.

Another important feature of the problems discussed in the class is that there is always a mathematical moral attached to the task. For example, the magic square problem (Can you place the numbers 1, 2, ..., 9 in a three by three box, so that the sum of each row, each column, and each diagonal is the same?) is a task in which students see and discuss the great power of the heuristic strategies. During the solution process of this problem, students discuss strategies that include establishing subgoals, working backwards, trying special cases, exploiting symmetry, making a systematic list, and exploring extreme cases. An important moral associated with the magic square problem is that what seems to be a trivial problem initially could produce interesting and even 'new' mathematical knowledge for the students. In addition, this problem was, later in the course, presented to the students in a different context. There, students were challenged to recognize and transfer their knowledge to other situations. The statement of the isomorphic problem to the magic square is:

Nine counters with the digits from 1 through 9 are placed on a table. Two players with the imaginative names of A and B take turns selecting counters from the table. The winner is the first player who has a set of counters including three counters whose sum is 15. (For example: If on the 9th pick you just picked up the 7 to give yourself the set 2, 3, 6, 7, 9, then you’ve won. Your opponent has 1, 4, 5, 8 and no three of those add up to 15—but your {2, 6, 7} do. The question: Does either player have a winning strategy?

Indeed, working on this problem led some students to note the same features of the magic square and its connection to the tic-tac-toe game (figure 1).

It is important to mention that it is common for the students to work on problems with different mathematical structures during a problem-solving class period. For example, the above problem (counters problem) was addressed by the students after they had worked on a problem that involved the use of expected value to analyse the fairness of a game. An instructional principle here is that students should develop mathematical tools and abilities to identify what type of structure the problem holds, and then be able to access a set of strategies and resources to attack the problem. Thus, the students are aware that the ideas being discussed during the class or the previous problem do not necessarily guarantee that the specific approach that worked for one problem also works for the next one. In fact, this challenges the presentation of the material in textbooks (calculus textbooks) in which the students normally know that the material or problems given in the examples provide similar tools and directions to work on the proposed list of problems.

![Figure 1. Magic square seen as the tic-tac-toe game.](image-url)
Many of the problems that students work during the course offer aspects in which there is opportunity to explore the information of the problem, analyse different representations, and evaluate particular cases. During this process, they might re-state the problem, establish a specific connection, find a pattern, or develop a generalization of the problem. An important moral shown in several problems is that any mathematical conjecture that could emerge from the students’ exploration must be supported with mathematical arguments. For example, the following problem was discussed during one of the problem-solving sessions:

Suppose you pick \( n \) points on the boundary of a circle. You then draw all of the line segments that connect pairs of those points. If the points have been chosen so that no three of the segments intersect at the same point (that is, the circle is divided into the maximum possible number of regions), into how many regions is the circle divided?

An attractive way to approach this problem is to try special cases in order to find a pattern or general expression. Students had worked on problems where this strategy seems to work. For example, What is the sum of the coefficients of \((x + 1)^n\)? or How many subsets does a set of \( n \) elements have? were some problems solved by the students in previous sessions. Indeed, the moral of working those problems was that the students need to provide a mathematical argument (proof) to support their conjectures. Thus, with this background, the students worked on the ‘region problem’ and showed the cases detailed in figure 2.

<table>
<thead>
<tr>
<th># of Points (( n ))</th>
<th># of Regions</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
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<td>3</td>
<td>4</td>
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<td>4</td>
<td>8</td>
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<tr>
<td>5</td>
<td>16</td>
</tr>
</tbody>
</table>

Figure 2. Heuristic: trying to find a pattern.
Based on this work some students were ready to conjecture that the total of regions for \( n \) points could be expressed as \( 2^{n-1} \). However, the search for a mathematical argument to support the conjecture led students to examine other approaches to find the conjecture. The class discussion focused on analysing relationships between connecting lines, points of intersection and number of regions. It was observed that the maximum number of regions when connecting two points is obtained by placing the \( n \) points on the circle in such a way that at most two connecting lines intersect at the same point. Then the number of regions will be given by the expression:

\[
1 + \binom{n}{2} + \binom{n}{4}; \text{ where } \binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

During the class discussion, students have the opportunity to analyse particular cases and observe what happens to the number of regions when a point is added on the boundary.

Note that this expression has the same result as the expression \( 2^{n-1} \) for \( n = 1, 2, 3, 4, \) and 5.

Several points were addressed during the discussion of this problem, an important one is that 'the only way that [an expression] is going to be true is that you have to prove it.' The empirical work is a powerful tool to explore mathematical ideas, but the results must be supported with a mathematical proof. Indeed, making the connections between empirical and formal work plays an important role in the development of mathematics.

6. Attention to the solution process

Another important aspect that appears consistently during the implementation of problem-solving activities is that students should pay attention to the process involved in reaching the solution(s) of the problem. This challenges the idea that the main goal for students while working on a problem is to find the solution. Paying attention to the process gives students the opportunity to analyse and compare diverse qualities of methods of solutions, and to look for applications and extensions of the problem. Some features of the course in which the students examine the solution process of the problem include actions in which the students are aware of the following points.

The solution of a problem is an initial point to launch new mathematical ideas. Thus, students are encouraged to work on different types of problems and to search for connections and extensions of the original problem. An important part of the students' approaches is to conceptualize that finding a solution of a problem is just the beginning of a process in which they have the opportunity to think of other methods of solution, to pose more questions or related problems, to extend the problem by changing the original conditions, and to evaluate new relations among other contexts. Thus, other problems emerge and students spend time discussing the qualities of different approaches used to solve those problems. This approach challenges the idea that students normally work on problems that are given to them by their instructors and rarely have the opportunity to go beyond a specific solution. In this context, the students could also explore ways in which the statement of the problem is changed. That is, they may analyse what happens if
one or more parts of the original statement are contradicted. Brown and Walter [14] called this activity the ‘what-if-not’ strategy and have used it extensively in their courses.

The analysis of the quality of different approaches asks the students to compare and value aspects in which it is important to think of what methods are more efficient than others. This activity is present throughout the development of the course. It is common that students show approaches that include the use of particular and general methods. For instance, when the students were dealing with the problem ‘prove $|\tan(x) + \cot(x)| \geq 2$’, a first approach was to use trigonometric identities to transform the right into manageable terms to show that the inequality was true. That is, expressing the right side as

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{1}{\cos x \sin x} = \frac{2}{\sin 2x}$$

which is always greater or equal than 2. In addition, another more general approach to the problem was shown. That is, representing the left side as

$$|\tan (x) + \frac{1}{\tan (x)}| \geq 2 \text{ or } |z + \frac{1}{z}| \geq 2$$

Then, the inequality could be represented as $z^2 + 1 \geq 2z$ or $(z - 1)^2 \geq 0$. This approach appeals more as an abstract form to represent the original expression rather than the specific content of the problem. This general approach is what Polya calls ‘the inventor’s paradox’ in which a more general problem may on occasion be easier to solve than the given problem.

Another example in which the students have to re-organize the initial data and then introduce additional concepts (weights) in order to think of a method of solution is:

Nominations have been taken for mathematics teacher of the year. Five candidates have been nominated and an election has been conducted in which students voted by ranking their 1st choice, 2nd choice, 3rd choice, etc. The results of the balloting by the 55 students are shown below:

<table>
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<tr>
<th>#vot</th>
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<th>12</th>
<th>10</th>
<th>9</th>
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<tr>
<td>1st</td>
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<td>c</td>
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Who is the winner?

In the same direction, homework assignments include problems in which students have the opportunity to discuss differences and qualities of each approach used to solve them. So a typical written assignment requires not only that students have to search for multiple ways to solve the tasks, but they have to write their ideas clearly and support their methods with mathematical arguments. Here, the students get to see that mathematics is not a finished product but a discipline in which there is room to find new developments. In fact, students’ verbal and
written communication play an important role in the discussion of limitations and extensions of mathematical ideas.

An example of a problem given to the students in their written assignment was:

How many rectangles are there on an 8 × 8 chess board? Be careful to count them all—any rectangle with its sides as grid lines in the chess board 'counts'.

As an example, note that the 2 × 2 chess board (figure 3).

As was mentioned above, an important part of the problem-solving instruction is that students are encouraged to communicate their ideas not just orally but also in a written form. While working on the homework problems, the students are asked to approach the problems in different ways and to write about the qualities of the approaches. Some of the ways that they used to achieve the solution are detailed below.

The use of special cases. An initial approach might involve trying to find a pattern by looking at special cases. So the problem is broken into manageable steps that allow one to count the number of rectangles for some particular cases. For example, focusing on how many rectangles there are in a 1 × n board, which leads to the expression \(1 + 2 + 3 + 4 + \ldots + n = \frac{n(n + 1)}{2}\). Similarly, counting the number of rectangles that there are in a 1 × m board and combining both results leads to the solution of the problem \(\left[\frac{(n(n + 1)/2)}{((m(m + 1)/2))}\right]\).

The use of coordinates. Another approach to solve this task involves the use of a coordinate system to determine the number of rectangles. That is, in the figure below, the total number of rectangles is the sum of \(i(1 + 2 + 3 + \ldots + 8)\) with \(i\) going from 1 to 8. That is, the sum of all the columns is \((1 + 2 + \ldots + 8)^2\). Or, for the general case an \(n \times m\) grid, the same reasoning can be applied to arrive at \(\left[\frac{(n(n + 1)/2)}{((m(m + 1)/2))}\right]\):

\[
\begin{array}{cccccccc}
8 \times 1 & 7 \times 1 & 6 \times 1 & 5 \times 1 & 4 \times 1 & 3 \times 1 & 2 \times 1 & 1 \times 1 \\
8 \times 2 & 7 \times 2 & 6 \times 2 & 5 \times 2 & 4 \times 2 & 3 \times 2 & 2 \times 2 & 1 \times 2 \\
8 \times 3 & 7 \times 3 & 6 \times 3 & 5 \times 3 & 4 \times 3 & 3 \times 3 & 2 \times 3 & 1 \times 3 \\
8 \times 4 & 7 \times 4 & 6 \times 4 & 5 \times 4 & 4 \times 4 & 3 \times 4 & 2 \times 4 & 1 \times 4 \\
8 \times 5 & 7 \times 5 & 6 \times 5 & 5 \times 5 & 4 \times 5 & 3 \times 5 & 2 \times 5 & 1 \times 5 \\
8 \times 6 & 7 \times 6 & 6 \times 6 & 5 \times 6 & 4 \times 6 & 3 \times 6 & 2 \times 6 & 1 \times 6 \\
8 \times 7 & 7 \times 7 & 6 \times 7 & 5 \times 7 & 4 \times 7 & 3 \times 7 & 2 \times 7 & 1 \times 7 \\
8 \times 8 & 7 \times 8 & 6 \times 8 & 5 \times 8 & 4 \times 8 & 3 \times 8 & 2 \times 8 & 1 \times 8
\end{array}
\]

The use of combinations. Yet, another approach could involve straight combinations of vertical and horizontal grid lines. That is, a rectangle can be chosen by
identifying two horizontal grid lines (top and bottom) and two vertical grid lines (left and right sides). Thus, when considering the $n \times m$ case, there are $n + 1$ vertical and $m + 1$ horizontal lines, so the total number of ways you can choose two horizontal and two vertical is given by:

$$\frac{n + 1}{2} \times \frac{m + 1}{2} = \frac{n(n + 1)}{2} \times \frac{m(m + 1)}{2}$$

The approaches shown above are a sample of how a homework task is expected to be done by the students. It is clear that they are asked to do more than just find the solution of a problem. Brown [15] pointed out that extending the problem to other domains is an important step in helping students to think of their own problems. It seems that the work done by the students outside of the classroom is an extension of what they normally do during the instruction. It is important to mention that the written assignments presented by the students are carefully reviewed by the instructor. In addition, the solutions proposed by the students are discussed with the whole class. This students' work is part of the evaluation they will receive at the end of the course. Other components of the final evaluation are the students' participation during the class and two written exams.

7. Developing a sense of confidence in students' behaviour

The development of mathematical disposition seems to be an important aspect of the problem-solving course. During the class development, students are encouraged to present their ideas to the class. They are aware that a clear mathematical argument is what counts in their presentation. In the discussion of their ideas, they expect criticism and challenges from other students. These activities are part of the process of dealing with any mathematical tasks. "... becoming a good thinker in any domain may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge" [16, p. 58].

A class environment in which students are constantly asked to explain and communicate their ideas to other students is an important feature of the class. For example, students work in small groups of three or four students during a significant portion of the class. The small groups are formed randomly and during the session, Schoenfeld makes sure that the students' interactions involve all the participants. Questions that help to frame the students' interactions include: What are you doing? Why are you doing that? and Where will that lead you? which are similar in spirit to those that Halmos [11] asks when teaching: What is true? What do the examples we can look at suggest? and How can it be done? The discussion of these questions encourages students to elaborate on what they are thinking, organize their ideas, and provide convincing arguments to defend their conjectures. Thus, students' ideas are normally challenged during the students' interaction based on examining other ways to solve a task, analysing connections, or refuting a counterexample. In addition, there is a set of expressions that become part of the classroom culture. For example, are we done?, do you know a related problem?, can you think of a special case?, can this be solve geometrically?, etc. are questions that appear while working on any task.

If you understand how things fit together in mathematics, there is very little to memorize. That is, the important thing in mathematics is to see
connections and see what makes things tick and how they fit together. Doing the mathematics is putting together the connections and making sense of the structure. Writing down the results—the formal statement that codify your understanding—is the end product, rather than the starting place [17, p. 328].

Students are exposed to the challenge of explaining why their ideas might work while dealing with the tasks. Comments and feedback that Schoenfeld provides to the students often involve examples in which the students have to rely on their own mathematical arguments to support their work. That is, the students should not expect the instructor to give the final word about the correctness of specific results; rather, the students have to construct and present their arguments to the rest of the class for discussion and judgment. A typical Schoenfeld’s response when a student asks for his approval to a mathematical work is:

Don’t look to me for approval, because I’m not going to provide it. I’m sure the class knows more than enough to say whether what’s on the board is right. So (turning to class) what do you folks think? [4, p. 62].

8. **Important phases in mathematical problem-solving instruction**

As was mentioned earlier students spend a significant part of the instruction time working in small groups. During this interaction they present and discuss their ideas about how to approach the problems. Here it is important to mention that the instructor constantly monitors the work done by each small group. During this process, the instructor frequently asks some clarification questions or challenges some of the approaches selected by the students. It seems that working in small groups helps students to discuss and defend their ideas related to the solution process and also provides useful information for the instructor on what type of information the students bring to the problem. Indeed, when later the students are asked to present their solution to the whole class, it is common that the instructor addresses some of the issues that arise from the small group discussions.

Schoenfeld has spent a great amount of time selecting, formulating, and redesigning the problems to be used during instruction. The problems illustrate the use of different mathematical ideas and in general are intended to show some mathematical ‘moral’ or messages. For example, the use of heuristics, the need to think of organized mathematical arguments, and the need to monitor the solution process are some aspects that constantly appear during the process of dealing with problems posed in different contexts. Thus, the students have the opportunity to reflect on and evaluate the potential of considering these types of ideas in their approaches to problems.

There are four main instructional activities that appear consistently throughout the course. Each one contributes to the development of students’ mathematical dispositions and plays an important role in their learning of mathematics. The instructional activities involve lectures, small group discussions, students presentations, and whole class discussions.

Schoenfeld sometimes gives lectures to the whole class. An important part here is the illustration of ‘real moves’ related to a specific approach used to solve a particular problem. That is, during the lecture, Schoenfeld tries to illustrate the different phases shown during the solution process including false starts and ways
to overcome them. So instead of presenting published mathematical knowledge, he gives special emphasis to the discussion of strategies that could help one recover from an unsuccessful approach (metacognitive activity). Some of the problems addressed by Schoenfeld are new to him and in his presentation, he tries to show what ideas could be useful, how to evaluate some approaches, and what makes him change or consider other approaches. Students are also required to study other materials such as the work of Polya and study some worked problems (learning by examples). During the lecture, the students are encouraged to participate and they are also encouraged to provide some problems for class discussion.

During the development of the course, students work on a variety of problems that demand the application of various types of strategies and mathematical knowledge. A significant part of the course is spent on the discussion of problems in small groups of students. Thus, they become active learners who are constantly exploring, conjecturing, and evaluating their ideas. It is common for students to rediscover or develop some mathematical results while working on some specific problems. ‘By virtue of participation... my students will develop a particular sense of the mathematical enterprise. The means are social, for the approach is grounded in the assumption that people develop their values and beliefs largely as a result of social interactions’ [4, p. 61].

Students are asked to present their ideas to the class. Here, they learn to present and communicate their ideas via some mathematical argument. They are aware that it is important to give a convincing explanation and rationale for their approaches. It is common that during the presentation, there will be examples or counterexamples that challenge the ideas or approaches they present. In fact, the main part of the students’ presentation is having to convince the rest of the class that they have solved the problem or made progress towards the solution.

Another instructional variant appears when the whole class approaches a specific problem. Here, the instructor coordinates the students’ ideas and often writes down what students consider to be the fundamental ideas related to the problem solution. In addition, the instructor may challenge some of the students’ ideas and as a consequence foster student participation.

It is important to mention that students receive feedback at different stages during the development of the problem-solving class. For example, during the small-group discussion, the idea(s) proposed by a student is discussed and evaluated within the small group. Later, when students present their approaches to the whole class there is also feedback from other students and the instructor. This feedback is an important instructional component that appears throughout the entire course.

It is important to note that, in general, the problems discussed during the class are used as a springboard for exploration of mathematical thinking. That is, students are constantly asked to think ahead in order to make connections or extensions to the problems.

9. Final remarks

A course based on problem solving, as discussed in this paper, includes several instructional activities that are identified as crucial in the learning of mathematics. It appears that in order to develop the students’ mathematical disposition to learn mathematics is important to provide a class environment in which students
consistently are asked to (a) work on tasks that offer diverse challenges; (b) discuss
the importance of using diverse types of strategies including the metacognitive
strategies; (c) participate in small and whole group discussions; (d) reflect on
feedback and challenges that emerge from interactions with the instructor and
other students; (e) communicate their ideas in written and oral forms; and (f) search
for connections and extensions of the problems. These learning activities play a
crucial role in helping students to see mathematics as a dynamic discipline in
which they have the opportunity to engage in mathematical discussions and thus
value the practice of doing mathematics.

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References

[1] Schoenfeld, A., 1988, Problem solving in context(s). In The Teaching and Assessing of
Mathematical Problem Solving, edited by R. J. Charles and E. A. Silver (Reston VA:
The National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence
Erlbaum Associates), pp. 82–92.
NJ: Lawrence Erlbaum Associates), pp. 53–70.
in the mathematics curriculum. In The Teaching and Assessing of Mathematical
Problem Solving, edited by R. I. Charles and E. A. Silver (Reston VA: The National
Council of Teachers of Mathematics), pp. 1–22.
Standards for School Mathematics (Reston, VA: NCTM).
via problem solving. In New Directions for Elementary School Mathematics, edited by
P. R. Trafton (Reston, VA: National Council of Teachers of Mathematics), pp. 31–
56.
Franklin Institute Press).
Constructing Mathematical Knowledge: Epistemology and Mathematics Education,

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